Homogeneous spacetimes and separation of variables in the Hamilton–Jacobi equation

2006 J. Phys. A: Math. Gen. 39 6641

(http://iopscience.iop.org/0305-4470/39/21/S64)

J. Phys. A: Math. Gen. **39** (2006) 6641–6647 doi:10.1088/0305-4470/39/21/S64

Homogeneous spacetimes and separation of variables in the Hamilton–Jacobi equation

Konstantin E Osetrin[1], **Valery V Obukhov**[1] **and Altair E Filippov**[2]

[1] Department of Theoretical Physics, Tomsk State Pedagogical University, Tomsk 634041, Russia
[2] Department of Mathematics, Tomsk State Pedagogical University, Tomsk 634041, Russia

E-mail: osetrin@tspu.edu.ru, obukhov@tspu.edu.ru and altair@tspu.edu.ru

Received 2 December 2005, in final form 29 January 2006
Published 10 May 2006
Online at stacks.iop.org/JPhysA/39/6641

Abstract
We discuss a classification of spatially homogeneous nonisotropic spacetimes, which admit a separation of variables in the Hamilton–Jacobi equation. Spacetimes with a four-parametric group of motion are considered. All types of metrics and Killing vectors of these spacetimes with Bianchi classification are obtained. An example of application for the Vaidya problem is considered.

PACS numbers: 04.20.Gz, 02.40.−k

1. Introduction

Let us consider a problem of classification of space-homogeneous models of spacetimes which admit a complete separation of variables in the Hamilton–Jacobi equation

$$g^{ij} S_{,i} S_{,j} = m^2, \qquad i, j, k = 1, \ldots, n. \tag{1}$$

These spaces are called Stäckel spaces [1–3]. Note that other important equations of motion (Klein–Gordon, Dirac, Weyl) can be integrated using the method of complete separation of variables only for metrics belonging to the class of the Stäckel spaces.

One of the main problems of mathematical physics for the gravity theory is the problem of exact integration of the field equations or equations of motion of matter. The most interesting models for cosmology are space-homogeneous models, which are known to admit a three-parametrical transitive group of motions with space-like orbits [5]. On the other hand, it is known [2] that geodesic equations in the Hamilton–Jacobi form can be integrated by the complete separation of variables method if they admit first integrals which are linear and quadratic with respect to the momenta

$$\underset{\nu}{X} = \underset{\nu}{X}^{ij} p_i p_j, \qquad \underset{p}{Y} = \underset{p}{Y}^{i} p_i. \tag{2}$$

These integrals commute pairwise with respect to the Poisson bracket and

$$\underset{\nu}{X}_{(ij;k)} = \underset{p}{Y}_{(i;j)} = 0 \tag{3}$$

(the semicolon denotes the covariant derivative and the parentheses denote symmetrization). Therefore ${Y_p}^i$, ${X_\nu}^{ij}$ are the components of the vector and tensor Killing fields, respectively.

Thus, there is a problem of finding a subclass of homogeneous spacetimes admitting complete sets of integrals of motion. In other words, a spacetime with a complete set must admit a three-parametrical transitive group of motions with space-like orbits. There are seven types of Stäckel spaces with the signature $(-, +, +, +)$. Let us consider Stäckel spacetimes of type (3.1). This type of Stäckel spacetimes is rather interesting in this context, because its metric depends only on such variable of a privileged coordinate set, which corresponds to null (wave) hypersurface of the Einstein equation. In other words, the Stäckel spacetimes of this type are common to spaces filled with radiation (gravitational, electromagnetic, etc).

2. Classification of the homogeneous Stäckel spacetimes (3.1) type

In a privileged coordinate set, where equation (1) admits variable separation, a metric of the (3.1) type has the form (see, for example, [4] and references therein)

$$g^{ij} = \begin{pmatrix} 0 & 1 & b_2(x^0) & b_3(x^0) \\ 1 & 0 & 0 & 0 \\ b_2(x^0) & 0 & a_{22}(x^0) & a_{23}(x^0) \\ b_3(x^0) & 0 & a_{23}(x^0) & a_{33}(x^0) \end{pmatrix}, \qquad a_{22}a_{33} > 0, \tag{4}$$

where x^0 is the wave-like variable.

The Killing equations for vector ξ^i and metric (4) have the following form:

$$\begin{aligned}
&1.\quad {\xi^0}_{,1} + b_2{\xi^0}_{,2} + b_3{\xi^0}_{,3} = 0\\
&2.\quad {\xi^0}_{,0} + {\xi^1}_{,1} + b_2{\xi^1}_{,2} + b_3\xi_{,3} = 0\\
&3.\quad b_2{\xi^0}_{,0} + {\xi^2}_{,1} + a_{22}{\xi^0}_{,2} + a_{23}{\xi^0}_{,3} + b_2{\xi^2}_{,2} + b_3{\xi^2}_{,3} - \xi^0 b_2' = 0\\
&4.\quad b_3{\xi^0}_{,0} + {\xi^3}_{,1} + a_{23}{\xi^0}_{,2} + a_{33}{\xi^0}_{,3} + b_2{\xi^3}_{,2} + b_3{\xi^3}_{,3} - \xi^0 b_3' = 0\\
&5.\quad {\xi^1}_{,0} = 0\\
&6.\quad {\xi^2}_{,0} + a_{22}{\xi^1}_{,2} + a_{23}{\xi^1}_{,3} = 0\\
&7.\quad {\xi^3}_{,0} + a_{23}{\xi^1}_{,2} + a_{33}{\xi^1}_{,3} = 0\\
&8.\quad b_2{\xi^2}_{,0} + a_{22}{\xi^2}_{,2} + a_{23}{\xi^2}_{,3} - \xi^0 a_{22}' = 0\\
&9.\quad b_3{\xi^2}_{,0} + b_2{\xi^3}_{,0} + a_{23}({\xi^2}_{,2} + {\xi^3}_{,3}) + a_{22}{\xi^3}_{,2} + a_{33}{\xi^2}_{,3} - \xi^0 a_{23}' = 0\\
&10.\quad b_3{\xi^3}_{,0} + a_{23}{\xi^3}_{,2} + a_{33}{\xi^3}_{,3} - \xi^0 a_{33}' = 0.
\end{aligned} \tag{5}$$

Metric (4) admits three commuting Killing vectors

$$X_1, X_2, X_3; \qquad [X_p, X_q] = 0, \qquad p, q, r = 1, 2, 3, \tag{6}$$

with components

$${X_p}^i = \delta^i_p. \tag{7}$$

The metric projection on orbits of this group of motions is degenerated. Thus, we need an additional Killing vector

$$X^i_4 = \xi^i. \tag{8}$$

The commutative relations of group X_1–X_4 have the form

$$\begin{aligned}
&[X_m, X_4] = \alpha_m X_4 + {\beta_m}^n X_n\\
&[X_1, X_4] = \alpha_1 X_4 + {\beta_1}^p X_p, \qquad p, q = 1, 2, 3\\
&[X_p, X_q] = 0.
\end{aligned} \tag{9}$$

This allows us to derive ξ^i,

$$\xi^i{}_{,1} = \alpha_1 \xi^i + \beta_1{}^p \delta_p{}^i, \qquad \xi^i{}_{,m} = \alpha_m \xi^i + \beta_m{}^n \delta_n{}^i. \tag{10}$$

The Jacobi identities for the structure constants (9) have the form

$$\beta_1{}^1 \alpha_m = 0, \qquad \alpha_2 \beta_3{}^n = \alpha_3 \beta_2{}^n, \qquad \alpha_m \beta_1{}^n = \alpha_1 \beta_m{}^n. \tag{11}$$

We can simplify equations (10) by using linear transformations of Killing vectors, which do not break the group structure.

(1) $X_m = S_m{}^n \tilde{X}_n$. Then equations (9) can be written in the form

$$[\tilde{X}_n, X_4] = \left(S_n{}^m\right)^{-1} \alpha_m X_4 + \left(S_n{}^l\right)^{-1} \beta_l{}^k S_k{}^m \tilde{X}_m$$
$$[X_1, X_4] = \alpha_1 X_4 + \beta_1{}^n S_n{}^m \tilde{X}_m + \beta_1{}^1 X_1.$$

Therefore, we have from (9) new structure constants

$$\tilde{\alpha}_1 = \alpha_1, \qquad \tilde{\alpha}_n = \left(S_n{}^m\right)^{-1} \alpha_m, \qquad \tilde{\beta}_1{}^1 = \beta_1{}^1,$$
$$\tilde{\beta}_1{}^m = \beta_1{}^n S_n{}^m, \quad \tilde{\beta}_n{}^m = \left(S_n{}^l\right)^{-1} \beta_l{}^k S_k{}^m. \tag{12}$$

(2) $X_4 = \tilde{X}_4 + b^m X_m$

$$[X_m, \tilde{X}_4] = \alpha_m \tilde{X}_4 + \left(\alpha_m b^n + \beta_m{}^n\right) X_n$$
$$[X_1, \tilde{X}_4] = \alpha_1 \tilde{X}_4 + \left(\alpha_1 b^n + \beta_1{}^n\right) X_n + \beta_1{}^1 X_1$$

and new constants

$$\tilde{\alpha}_p = \alpha_p, \qquad \tilde{\beta}_1{}^1 = \beta_1{}^1, \qquad \tilde{\beta}_p{}^m = \beta_p{}^m + \alpha_p b^m. \tag{13}$$

We also use coordinate transformations, which do not break the form of metric (4),

$$\tilde{x}^0 = a^0 x^0, \qquad \tilde{x}^1 = \frac{1}{a^0} x^1, \qquad \tilde{x}^n = a_p{}^n x^p. \tag{14}$$

From commutative relations (10) and Jacobi identities we obtain the following form of Killing vector:

$$\xi^i = \beta_p{}^q \delta_q{}^i x^p + f^i(x^0), \qquad \beta_m{}^1 = 0. \tag{15}$$

The Killing equations for this vector have the form

$$\begin{aligned}
&1. \quad \beta_1{}^1 + f^{0\prime} = 0 \\
&2. \quad \beta_1{}^2 + \beta_2{}^2 b_2 + \beta_3{}^2 b_3 + f^{0\prime} b_2 - f^0 b_2' = 0 \\
&3. \quad \beta_1{}^3 + \beta_2{}^3 b_2 + \beta_3{}^3 b_3 + f^{0\prime} b_3 - f^0 b_3' = 0 \\
&4. \quad f^{1\prime} = f^{2\prime} = f^{3\prime} = 0 \\
&5. \quad 2\beta_2{}^2 a_{22} + 2\beta_3{}^2 a_{23} - f^0 a_{22}' = 0 \\
&6. \quad \beta_2{}^3 a_{22} + \left(\beta_2{}^2 + \beta_3{}^3\right) a_{23} + \beta_3{}^2 a_{33} - f^0 a_{23}' = 0 \\
&7. \quad 2\beta_2{}^3 a_{23} + 2\beta_3{}^3 a_{33} - f^0 a_{33}' = 0,
\end{aligned} \tag{16}$$

and we immediately obtain

$$f^0 = -\beta_1{}^1 x^0 + \sigma^0, \qquad f^1 = \sigma^1, \qquad f^2 = f^3 = 0, \qquad \sigma = \text{const}.$$

Further classification can be made by use of the following two independent conditions:

1. According to the value of ${\beta_1}^1$. If ${\beta_1}^1 \neq 0$, then we may choose ${\beta_1}^1 = 1, \sigma^0 = \sigma^1 = 0$; if ${\beta_1}^1 = 0$, then we may choose $\sigma^0 = \sigma^1 = 1$. Thus, we have two types
   Type A: $\xi^0 = -x^0, \xi^1 = x^1$;
   Type B: $\xi^0 = 1, \xi^1 = 1$.
2. According to the classes of matrix ${\beta_m}^n$. By use of transformation (12) it can be brought to one of the following three classes:

$$(1)\ \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_3 \end{pmatrix} \quad (2)\ \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \quad (3)\ \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}, \qquad \beta \neq 0. \qquad (17)$$

   Finally, we have six classes A1, A2, A3, B1, B2, B3 with some subclasses, which can appear along Killing equations solving.

Type A spaces

Class A1. We have here three subclasses dependent on constants λ_m.

Subclass A1.1

$$g^{ij} = \begin{pmatrix} 0 & 1 & \beta_2 {x^0}^{1-\lambda_2} & \beta_3 {x^0}^{1-\lambda_3} \\ 1 & 0 & 0 & 0 \\ \beta_2 {x^0}^{1-\lambda_2} & 0 & {x^0}^{-2\lambda_2} & \alpha_{23} {x^0}^{-\lambda_2-\lambda_3} \\ \beta_3 {x^0}^{1-\lambda_3} & 0 & \alpha_{23} {x^0}^{-\lambda_2-\lambda_3} & {x^0}^{-2\lambda_3} \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} -x^0 \\ x^1 \\ \lambda_2 x^2 \\ \lambda_3 x^3 \end{pmatrix},$$

$$\lambda_2 \neq 1, \qquad \lambda_3 \neq 1, \qquad |\alpha_{23}| < 1.$$

If $\lambda_2 = \lambda_3$, then we may choose $\alpha_{23} = 0$.

The Bianchi classification of this metric looks like the following: type I for $\lambda_2 = \lambda_3 = 0$, type V for $\lambda_2 = \lambda_3 \neq 0$, type VI for $\lambda_2 \neq \lambda_3$.

The scalar curvature (R) and the Weyl tensor (W) of this metric are

1. $R = \text{const} \neq 0$ for $\lambda_2 = \lambda_3 = 0$;
2. $R = 0$ for $\beta_2 = \beta_3 = 0$;
3. $W = 0$ for $\beta_2 = \beta_3 = 0$, $\lambda_2 = \lambda_3 \neq 0$ (conformally flat space).

Subclass A1.2

$$g^{ij} = \begin{pmatrix} 0 & 1 & -{\beta_1}^2 \log x^0 & \beta_3 {x^0}^{1-\lambda_3} \\ 1 & 0 & 0 & 0 \\ -{\beta_1}^2 \log x^0 & 0 & \frac{1}{{x^0}^2} & \alpha_{23} {x^0}^{-1-\lambda_3} \\ \beta_3 {x^0}^{1-\lambda_3} & 0 & \alpha_{23} {x^0}^{-1-\lambda_3} & {x^0}^{-2\lambda_3} \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} -x^0 \\ x^1 \\ {\beta_1}^2 x^1 + x^2 \\ \lambda_3 x^3 \end{pmatrix},$$

$$\lambda_3 \neq 1, \qquad |\alpha_{23}| < 1.$$

The Bianchi classification of this metric is of type VI only.

Subclass A1.3

$$g^{ij} = \begin{pmatrix} 0 & 1 & -{\beta_1}^2 \log x^0 & -{\beta_1}^3 \log x^0 \\ 1 & 0 & 0 & 0 \\ -{\beta_1}^2 \log x^0 & 0 & \frac{1}{{x^0}^2} & 0 \\ -{\beta_1}^3 \log x^0 & 0 & 0 & \frac{1}{{x^0}^2} \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} -x^0 \\ x^1 \\ {\beta_1}^2 x^1 + x^2 \\ {\beta_1}^2 x^1 + x^3 \end{pmatrix}.$$

The Bianchi classification of this metric is of type V only. The scalar curvature: $R = \text{const} > 0$.

Class A2. We have two subclasses dependent on constant λ.

Subclass A2.1

$$g^{ij} = \begin{pmatrix} 0 & 1 & \beta_2 {x^0}^{1-\lambda} & (\beta_3 - \beta_2 \log x^0){x^0}^{1-\lambda} \\ 1 & 0 & 0 & 0 \\ \beta_2 {x^0}^{1-\lambda} & 0 & {x^0}^{-2\lambda} & -{x^0}^{-2\lambda} \log x^0 \\ (\beta_3 - \beta_2 \log x^0){x^0}^{1-\lambda} & 0 & -{x^0}^{-2\lambda} \log x^0 & (\alpha_{33} + \log^2 x^0){x^0}^{1-\lambda} \end{pmatrix}$$

$$\xi^i = (-x^0, x^1, \lambda x^2, x^2 + \lambda x^3), \qquad \lambda \neq 1$$

The Bianchi classification of this metric looks like the following: type II for $\lambda = 0$, type IV for $\lambda \neq 0$.

The scalar curvature $R = 0$ for $b_2 = b_3 = 0$.

Subclass A2.2

$$\begin{aligned} b_2 &= -{\beta_1}^2 \log x^0 \\ b_3 &= -{\beta_1}^3 \log x^0 + \tfrac{1}{2}{\beta_1}^2 \log^2 x^0 \\ a_{22} &= 1/{x^0}^2 \\ a_{23} &= -\log x^0/{x^0}^2 \\ a_{33} &= (\alpha_{33} + \log^2 x^0)/{x^0}^2 \\ \xi^i &= (-x^0, x^1, {\beta_1}^2 x^1 + x^2, {\beta_1}^3 x^1 + x^2 + x^3), \qquad \alpha_{33} > 0. \end{aligned}$$

The Bianchi classification of this metric is of type IV only. The scalar curvature $R = \text{const} \geqslant 0$, $R = 0$ for $b_2 = b_3 = 0$.

Class A3. We may choose $\beta = 1$; therefore, the solution is as follows:

$$\begin{aligned} b_2 &= (\beta_2 \cos t + \beta_3 \sin t){x^0}^{1-\alpha} \\ b_3 &= (-\beta_2 \sin t + \beta_3 \cos t){x^0}^{1-\alpha} \\ a_{22} &= (\alpha_{23} \sin 2t + \alpha_{22} \cos 2t + \alpha_{33}){x^0}^{-2\alpha} \\ a_{23} &= (-\alpha_{22} \sin 2t + \alpha_{23} \cos 2t){x^0}^{-2\alpha} \\ a_{33} &= (-\alpha_{23} \sin 2t - \alpha_{22} \cos 2t + \alpha_{33}){x^0}^{-2\alpha} \\ t &= \log x^0. \end{aligned} \quad , \qquad \xi^i = \begin{pmatrix} -x^0 \\ x^1 \\ \alpha x^2 - x^3 \\ x^2 + \alpha x^3 \end{pmatrix},$$

The Bianchi classification of this metric is of type VII only.

Type B spaces

Class B1. This class has three subclasses.

Subclass B1.1

$$g^{ij} = \begin{pmatrix} 0 & 1 & \beta_2\, e^{\lambda_2 x^0} & \beta_3\, e^{\lambda_3 x^0} \\ 1 & 0 & 0 & 0 \\ \beta_2\, e^{\lambda_2 x^0} & 0 & e^{2\lambda_2 x^0} & \alpha_{23}\, e^{(\lambda_2+\lambda_3)x^0} \\ \beta_3\, e^{x^0} & 0 & \alpha_{23}\, e^{(\lambda_2+\lambda_3)x^0} & e^{2\lambda_3 x^0} \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} 1 \\ 1 \\ \lambda_2 x^2 \\ \lambda_3 x^3 \end{pmatrix},$$

$$\lambda_2 \neq 0, \qquad \lambda_3 \neq 0, \qquad |\alpha_{23}| < 1$$

If $\lambda_2 = 1$, we may choose $\alpha_{23} = 0$.

The Bianchi classification of this metric looks like the following: type V for $\lambda_2 = \lambda_3 \neq 0$, type VI for $\lambda_2 \neq \lambda_3$.

Subclass B1.2

$$g^{ij} = \begin{pmatrix} 0 & 1 & {\beta_1}^2 x^0 & \beta_3\, e^{\lambda_3 x^0} \\ 1 & 0 & 0 & 0 \\ {\beta_1}^2 x^0 & 0 & 1 & \alpha_{23}\, e^{\lambda_3 x^0} \\ \beta_3\, e^{\lambda_3 x^0} & 0 & \alpha_{23}\, e^{x^0} & e^{2\lambda_3 x^0} \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} 1 \\ 1 \\ {\beta_1}^2 x^1 \\ \lambda_3 x^3 \end{pmatrix},$$

$$\lambda_3 \neq 0, \qquad |\alpha_{23}| < 1.$$

The Bianchi classification of this metric is of type III only.

Subclass B1.3

$$g^{ij} = \begin{pmatrix} 0 & 1 & {\beta_1}^2 x^0 & {\beta_1}^3 x^0 \\ 1 & 0 & 0 & 0 \\ {\beta_1}^2 x^0 & 0 & 1 & 0 \\ {\beta_1}^3 x^0 & 0 & 0 & 1 \end{pmatrix}, \qquad \xi^i = \begin{pmatrix} 1 \\ 1 \\ {\beta_1}^2 x^1 \\ {\beta_1}^3 x^3 \end{pmatrix}.$$

The Bianchi classification of this metric is of type I.

Class B2

Subclass B2.1

$$g^{ij} = \begin{pmatrix} 0 & 1 & \beta_2\, e^{\lambda x^0} & (\beta_3 + \beta_2 x^0)\, e^{\lambda x^0} \\ 1 & 0 & 0 & 0 \\ \beta_2\, e^{\lambda x^0} & 0 & e^{2\lambda x^0} & x^0 e^{2\lambda x^0} \\ (\beta_3 + \beta_2 x^0)\, e^{\lambda x^0} & 0 & x^0\, e^{2\lambda x^0} & (\alpha_{33} + {x^0}^2)\, e^{2\lambda x^0} \end{pmatrix},$$

$$\xi^i = (1, 1, \lambda x^2, x^2 + \lambda x^3), \qquad \lambda \neq 0, \qquad \alpha_{33} \neq 0.$$

The Bianchi classification of this metric is of type IV.

Subclass B2.2

$$g^{ij} = \begin{pmatrix} 0 & 1 & {\beta_1}^2 x^0 & {\beta_1}^3 x^0 + \frac{1}{2}{\beta_1}^2 {x^0}^2 \\ 1 & 0 & 0 & 0 \\ {\beta_1}^2 x^0 & 0 & 1 & x^0 \\ {\beta_1}^3 x^0 + \frac{1}{2}{\beta_1}^2 {x^0}^2 & 0 & x^0 & \alpha_{33} + {x^0}^2 \end{pmatrix},$$

$$\xi^i = \left(1, 1, {\beta_1}^2 x^1, {\beta_1}^3 x^1 + x^2\right), \qquad \alpha_{33} \neq 0.$$

The Bianchi classification of this metric is of type II.

Class B3

$$\begin{aligned} b_2 &= (\beta_2 \cos x^0 + \beta_3 \sin x^0)\, e^{\alpha x^0} \\ b_3 &= (-\beta_2 \sin x^0 + \beta_3 \cos x^0)\, e^{\alpha x^0} \\ a_{22} &= (\alpha_{23} \sin 2x^0 + \alpha_{22} \cos 2x^0 + \alpha_{33})\, e^{2\alpha x^0} \\ a_{23} &= (-\alpha_{22} \sin 2x^0 + \alpha_{23} \cos 2x^0)\, e^{2\alpha x^0} \\ a_{33} &= (-\alpha_{23} \sin 2x^0 - \alpha_{22} \cos 2x^0 + \alpha_{33})\, e^{2\alpha x^0} \end{aligned} \qquad \xi^i = \begin{pmatrix} 1 \\ 1 \\ \alpha x^2 - x^3 \\ x^2 + \alpha x^3 \end{pmatrix}.$$

The Bianchi classification of this metric is of type VII.

Let us present the obtained results in a form of a table.

			Bianchi type								
Classes			I	II	III	IV	V	VI	VII	VIII	IX
A	1	A1.1	$\lambda_2, \lambda_3 = 0$				$\lambda_2 = \lambda_3$	$\lambda_2 \neq \lambda_3$			
		A1.2		$\lambda_3 = 0$				$\lambda_3 \neq 0$			
		A1.3				+					
	2	A2.1		$\lambda = 0$		$\lambda \neq 0$					
		A2.2				+					
	3	A3							+		
B	1	B1.1					$\lambda_2 = \lambda_3$	$\lambda_2 \neq \lambda_3$			
		B1.2			+						
		B1.3	+								
	2	B2.1				+					
		B2.2		+							
	3	B3							+		

Let us consider an example of application of obtained metrics in the Vaidya problem. The Vaidya problem is associated with high-frequency radiation of general nature (gravitational, electromagnetic, etc) and is most appropriate to demonstrate the obtained metrics. Einstein–Vaidya equations have the form [6, 7]

$$R_{\alpha\beta} - \tfrac{1}{2} R g_{\alpha\beta} - \Lambda g_{\alpha\beta} = q^2 l_\alpha l_\beta, \qquad l_\alpha l^\alpha = 0, \tag{18}$$

where Λ is the cosmological term, q is the energy density of radiation and l_α is the wave vector.

Let us assume for simplicity that $g_{02} = g_{03} = 0$. Then $l_\alpha = (l, 0, 0, 0)$, $\Lambda = 0$ and $R = 0$. It is easy to see that the obtained metrics satisfy equations (18) under the only condition for the energy density of radiation. For all our metrics, equations (18) resulting in only one equation

$$(ql)^2 = \frac{k_1}{(x^0)^2} + k_2, \tag{19}$$

where k_1 and k_2 are constants. For type A metrics we have $k_2 = 0$ and for type B metrics $k_1 = 0$. The obtained solution represents an analogue of spherical wave or an analogue of homogeneous radiation (for $k_1 = 0$) occupying the space (for 7 types of homogeneous spaces by Bianchi classification and for 12 types of obtained metrics).

References

[1] Stäckel P 1893 *Math. Ann.* **42** 537

[2] Obukhov V V and Osetrin K E 2004 *Proc. Sci.* WC2004 027 (pos.sissa.it)

[3] Bagrov V G, Obukhov V V, Osetrin K E and Filippov A E 1999 *Gravit. Cosmol.* **5** 10 (supplement)

[4] Bagrov V G, Obukhov V V and Shapovalov A V 1986 *Pramana J. Phys.* **26** 93–108

[5] Bogoyavlensky O I 1985 *Methods in the Qualitative Theory of Dynamical Systems in Astrophysics and Gas Dynamics* (Berlin: Springer) pp 301

[6] Vaidya P S 1951 *Phys. Rev.* **83** 10

[7] Isaacson R A 1968 *Phys. Rev.* **166** 1263